On the modified Selberg integral of the three-divisor function d_3

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Abstract. We prove a non-trivial result for the, say, modified Selberg integral $\widetilde{J}_3(N,h)$, of the divisor function $d_3(n) := \sum_a \sum_b \sum_{c,abc=n} 1$; this integral is a slight modification of the corresponding Selberg integral, that gives the expected value of the function in short intervals. We get, in fact, $\widetilde{J}_3(N,h) \ll Nh^2L^2$, where $L := \log N$; furthermore, as a byproduct, we obtain indications on the way in which it may be proved the weak sixth moment of the Riemann zeta function.

1. Introduction and statement of the results.

We study a kind of modification of the Selberg integral (see [C2]) of the arithmetic function $f : \mathbb{N} \to \mathbb{C}$, namely

$$J_f(N,h) \overset{def}{=} \int_{hN^{\varepsilon}}^{N} \Big| \sum_{x<n\le x+h} f(n) - M_f(x,h) \Big|^2 dx$$

is the Selberg integral of our f and $M_f(x,h)$ is the expected value in the short interval $[x, x+h]$.

As usual, $[x, x+h]$ is short whenever $h = o(x)$ (and $h \to \infty$ to avoid trivialities) as $x \to \infty$. Then, writing now on $x \sim N$ in sums to mean $N < x \le 2N$, the modified Selberg integral of the arithmetic function $f : \mathbb{N} \to \mathbb{C}$, namely (compare [C1] definitions)

$$\widetilde{J}_f(N,h) \overset{def}{=} \sum_{x\sim N} \Big| \sum_{0\le |n-x|\le h} \Big(1 - \frac{|n-x|}{h}\Big) f(n) - M_f(x,h) \Big|^2,$$

has the same mean-value $M_f(x,h)$ of the Selberg integral (and this is vital for the link with the moments of the Riemann zeta-function, compare §2).

We are interested, here, in the arithmetic functions d_k having generating Dirichlet series ζ^k, where ζ is the Riemann function; i.e., in the $k-$divisor functions, as these are naturally linked with the moments of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = 1/2$, as proved in [C2]; these $d_k(n)$ are defined elementarily as the number of ways to write n as a product of k positive integers. We will indicate for short their Selberg integral as

$$J_k(N,h) \overset{def}{=} \int_{hN^{\varepsilon}}^{N} \Big| \sum_{x<n\le x+h} d_k(n) - M_k(x,h) \Big|^2 dx$$

and their modified Selberg integral as

$$\widetilde{J}_k(N,h) \overset{def}{=} \sum_{x\sim N} \Big| \sum_{0\le |n-x|\le h} \Big(1 - \frac{|n-x|}{h}\Big) d_k(n) - M_k(x,h) \Big|^2,$$

where it is not a pure chance that the mean-values $M_k(x,h)$ are the same (compare §2 calculations and considerations).

In fact, here we will study the case $k = 3$, in particular, to show an estimate not for the Selberg integral of d_3, but for the modified Selberg integral of it. However, it turns out that the same link in [C2] with the Selberg integral of d_k should hold, more in general, for the modified Selberg integral of d_k, too. This general link will be provided in the future, as it will be fundamental for future approaches for the proof of the Lindelöf Hypothesis (LH, for short); but the present path working for $k = 3$ has obstructions for the cases $k > 3$ (due to $h-$range limitations, see Theorem 1.1 in [C2]).

Finally, we point out both that we may bound the integral $\widetilde{J}_k$ for all integers $k \ge 3$, too (even if this doesn't give any bound for $I_k(T)$, see [C2], we still haven't found square-root cancellation!); and that all these estimates are unreachable for the corresponding Selberg integral (at least, for the present methods !).

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Actually, for general arithmetic functions $f:\mathbb{N}\to\mathbb{C}$, we define the discrete variant above, since, at least for the "essentially bounded" f, the discrete and the continuous mean-squares are close to one another, see [C1]. For example, hereafter ESSENTIALLY BOUNDED is abbreviated $f \lll 1$ which means that $\forall\varepsilon>0$ we have $|f(n)| \ll_{\varepsilon} n^{\varepsilon}$; also, we write $A(N,h) \lll B(N,h)$ whenever $\forall\varepsilon>0$ we have $A(N,h) \ll_{\varepsilon} N^{\varepsilon}B(N,h)$, so

$$\int_N^{2N}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)d_3(n)-M_3(x,h)\Big|^2dx \lll \int_N^{2N}\Big|\sum_{0\le|n-[x]|\le h}\Big(1-\frac{|n-[x]|}{h}\Big)d_3(n)-M_3([x],h)\Big|^2dx$$

$$+N+\int_N^{2N}\Big|M_3(x,h)-M_3([x],h)\Big|^2dx \lll \sum_{N\le x<2N}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)d_3(n)-M_3(x,h)\Big|^2+N+h^2/N,$$

because $M_3(x,h)=hQ_2(\log x)$, where Q_2 is a quadratic polynomial (see §2 for details), whence

$$\frac{d}{dx}Q_2(\log x)=Q_2'(\log x)/x \ll \frac{\log x}{x}$$

and the mean-value theorem gives

$$M_3([x],h)=M_3(x,h)+O\left(\frac{h\log x}{x}\right).$$

Then, since the quantities in the square contribute $\lll h$ for the single terms $x=N$ and $x=2N$,

$$\int_N^{2N}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)-M_f(x,h)\Big|^2dx \lll \widetilde{J}_f(N,h)+N+h^2.$$

In the same way,

$$\widetilde{J}_f(N,h) \lll \int_N^{2N}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)-M_f(x,h)\Big|^2dx+N+h^2.$$

Remainders are negligible, compare next bounds in the following Theorem.

We give our main result.

THEOREM. *Let* $N,h\in\mathbb{N}$, *with* $h=h(N)\to\infty$ *and* $h=o(\sqrt{N})$ *when* $N\to\infty$. *Then* $\forall\varepsilon>0$

$$\sum_{x\sim N}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)d_3(n)-M_3(x,h)\Big|^2 \ll_{\varepsilon} N^{\varepsilon}(Nh+N^{2/3}h^2)+Nh^2L^2$$

and

$$\int_N^{2N}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)d_3(n)-M_3(x,h)\Big|^2dx \ll_{\varepsilon} N^{\varepsilon}(Nh+N^{2/3}h^2)+Nh^2L^2.$$

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The paper is organized as follows:

- in section 2 we calculate the mean-values of both the Selberg and the modified Selberg integral of d_3, give some remarks on the Cesaro averages we will perform both on the long/the short summations and define the Fourier coefficients for the local sums;
- in section 3 we give the Lemmas to prove a general Proposition from which we finally prove the Theorem.

2. The modified Selberg integral: mean values, Cesaro averages and Fourier coefficients.

First of all, we need to clarify how do we define the mean-values $M_f(x,h)$, both the same for the Selberg integral and for the modified Selberg integral.

This fact is vital for the link with the moments, on the critical line, of the Riemann zeta-function, compare [C2] results.

We expect the same mean-value for the following, elementary reason (now on $a \ge 1$ is implicit in $\sum_{a\le A}$):

$$\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)=\frac{1}{h}\sum_{m\le h}\sum_{0\le|n-x|<m}f(n)$$

so that

$$\sum_{0\le|n-x|<m}f(n)\approx M_f(x,2m-1)\quad\Rightarrow\quad\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)\approx\frac{1}{h}\sum_{m\le h}M_f(x,2m-1)$$

and at least, say, for SEPARABLE mean-values

$$M_f(x,t)=t\ \mathcal{M}_f(x)\quad\forall t=o(x)$$

where $\mathcal{M}_f(x)$ is a, say, SLOWLY VARYING function with respect to x, here, namely with a "small derivative", w.r.t. x (esp., polynomials in the variable $\log x$), we get

$$\sum_{0\le|n-x|<m}f(n)\approx M_f(x,2m-1)=(2m-1)\mathcal{M}_f(x)\quad\Rightarrow\quad\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)\approx h\mathcal{M}_f(x)=M_f(x,h).$$

This is true, at least for the noteworthy case of the functions $f=d_k$, see the following.

We say something general, for the modified Selberg integrals of essentially bounded, arithmetic functions (a.f.) $f:\mathbb{N}\to\mathbb{C}$. For this kind of generality, we need to define the ERATOSTHENES TRANSFORM of our f, namely, $f*\mu$, where $*$ is Dirichlet convolution and μ is the Möbius function. The Dirichlet convolution, write f_1*f_2, of the arithmetic functions f_1 and f_2 is defined as (see [Te] for details and properties)

$$(f_1*f_2)(n)\overset{def}{=}\sum_{d|n}f_1(d)f_2(n/d),\quad\forall n\in\mathbb{N}.$$

Notice that the Dirichlet convolution of bounded functions needs not be bounded : esp., $f_1=f_2=\mathbf{1}$, the CONSTANT ONE-function, gives $f_1*f_2=\mathbf{d}$, the DIVISOR FUNCTION, that is not bounded; however, it's essentially bounded (see [Te]) and, in fact, it's easy to prove that the Dirichlet convolution keeps the essential boundedness (f_1,f_2 ess.bounded $\Rightarrow f_1*f_2$ ess.bounded).

Here, we recall that $\mu(1)\overset{def}{=}1$ and $\mu(n)\overset{def}{=}(-1)^r$, $\forall n=p_1\cdots p_r$, i.e. on square-free numbers which are the product of r distinct primes; $\mu(n)\overset{def}{=}0$ exactly when n is not square-free (i.e., there's a prime p with $p^2|n$). The Möbius inversion formula states that $f=g*\mathbf{1}$ if and only if $g=f*\mu$ (whatever $f,g:\mathbb{N}\to\mathbb{C}$ are); hence, f is ess.bounded $\Leftrightarrow$ g is ess.bounded. In the following, we'll indicate usually the Eratosthenes transform of our f, i.e. $f*\mu$, as g.

Then, the modified Selberg integral of our $f(n)=\sum_{q|n}g(q)$ is

$$\widetilde{J}_f(N,h)=\widetilde{J}_{g*\mathbf{1}}(N,h)=\sum_{x\sim N}\Big|\sum_q g(q)\sum_{\substack{0\le|n-x|\le h\\ n\equiv0(\mathrm{mod}\ q)}}\Big(1-\frac{|n-x|}{h}\Big)-M_f(x,h)\Big|^2.$$

The issue of where to stop $q-$summation is not a trivial problem: if g has no support-limits, we may write $q \le x+h$, as $q > x+h$ gives an empty sum over n, see the above. However, the support of g may be of some help in "cutting the range of summation", for the $q-$sum. These q will be called the DIVISORS, of our arithmetic function f, whose RANGES are crucial for the forthcoming analysis.

In fact, we wish to write also the mean-value of our f, say $M_f(x,h)$, as a sum over these divisors. We'll say that we give an ARITHMETIC FORM of the mean-value $M_f(x,h)$, if we write it as

$$M_f(x,h) = h \sum_q \frac{g(q)}{q} + \text{good error},$$

while we give it an ANALYTIC FORM if we calculate it from residues with $f(n)$'s Dirichlet generating function.

We will not embark into this discussion for general essentially bounded functions, but only give a glimpse for the cases of our $f = d_k$, since

$$d_k(n) \overset{def}{=} \sum_{\substack{n_1 \\ n_1 \cdots n_k = n}} \cdots \sum_{n_k} 1 = \sum_{q|n} \sum_{\substack{n_1 \\ n_1 \cdots n_{k-1} = q}} \cdots \sum_{n_{k-1}} 1 = \sum_{q|n} d_{k-1}(q)$$

and the, say, $k-$FOLDING, in general, consists in writing (for problems in s.i., short intervals)

$$d_k(n) = \sum_{\substack{q|n \\ q \le Q}} g_k(q) + \text{good error},$$

where the other arithmetic function is a "good error", in the s.i., and $Q \ll x^{1-1/k}$, compare [C0] for an elementary approach to this problem; while we'll see only the case $k=3$ in the proof of the Theorem, §3.

However, we'll express first in analytic form our mean-value $M_3(x,h)$:

$$M_3(x,h) \overset{def}{=} h\, Q_2(\log x),$$

where

$$Q_2(t) \overset{def}{=} P_2(t) + P_2'(t) = \frac{t^2}{2} + 3\gamma t + (3\gamma^2 + 3\gamma_1), \quad \forall t \in \mathbb{R},$$

with

$$P_2(t) = \frac{t^2}{2} + (3\gamma - 1)t + (3\gamma^2 + 3\gamma_1 - 3\gamma + 1), \quad \forall t \in \mathbb{R},$$

which is

$$P_2(\log x) \overset{def}{=} \operatorname*{Res}_{s=1} \frac{x^{s-1}}{s} \zeta(s)^3,$$

from the Residues Theorem, applied for the d_3 summation formula (we confine to the elementary [Ti] error)

$$\sum_{n \le x} d_3(n) = x P_2(\log x) + O(x^{2/3} \log x), \quad \forall x \in \mathbb{N} \ (x \to \infty).$$

(Why $Q_2 = P_2 + P_2'$? We approximate $\sum_{x<n\le x+h} d_3(n)$ in J_3 from this formula and the mean-value th.m.) Here the elementary definitions of the EULER-MASCHERONI CONSTANT $\gamma \approx 0.577\ldots$ and the other are

$$\gamma \overset{def}{=} \lim_m \left(\sum_{j \le m} \frac{1}{j} - \log m \right), \quad \gamma_1 \overset{def}{=} -\lim_m \left(\sum_{j \le m} \frac{\log j}{j} - \frac{1}{2} \log^2 m \right).$$

(More in general, we may define the so-called STIELTJES CONSTANTS

$$\gamma_n \stackrel{def}{=} \frac{(-1)^n}{n!} \lim_m \left(\sum_{j \le m} \frac{\log^n j}{j} - \frac{\log^{n+1} m}{n+1} \right)$$

in order to express the mean-value for d_k, see [Ti], from $P_{k-1}(\log x) \stackrel{def}{=} \mathrm{Res}_{s=1} \frac{x^{s-1}}{s} \zeta(s)^k$. Thus, choosing $Q_k := P_k + P_k'$, $\forall k$, gives, see the above, the slowly varying function $\mathcal{M}_k(x) = Q_{k-1}(\log x)$ we need to say that the d_k mean-value in the s.i., i.e. $M_k(x,h) \stackrel{def}{=} h\, Q_{k-1}(\log x)$, is SEPARABLE, $\forall k \in \mathbb{N}$.).

After given an arithmetic form to $M_k(x,h)$ (compare Theorem proof, for the only case $k=3$), we get

$$\widetilde{J}_k(N,h) \ll \sum_{x \sim N} \Big| \sum_{q \le Q(x)} g_k(q) \widetilde{\chi}_q(x,h) \Big|^2 + \text{good},$$

where

$$\widetilde{\chi}_q(x,h) \stackrel{def}{=} \sum_{\substack{0 \le |n-x| \le h \\ n \equiv 0 (\bmod\ q)}} \left(1 - \frac{|n-x|}{h} \right) - \frac{h}{q}, \qquad \forall h, q \in \mathbb{N}, \ \ \forall x \in \mathbb{Z},$$

with $Q(x) = Q(x,N,h)$ depending on x, N and h, for a suitable g_k, see the above discussion on $k-$folding.

We can start our method from the ORTHOGONALITY OF ADDITIVE CHARACTERS abbreviating imaginary exponentials, as usual, $e(\alpha) \stackrel{def}{=} e^{2\pi i \alpha}$, $\forall \alpha \in \mathbb{R}$, and $e_q(a) \stackrel{def}{=} e(a/q)$, $\forall q \in \mathbb{N}$, $\forall a \in \mathbb{Z}$, to get

$$(0) \qquad \widetilde{\chi}_q(x,h) = \frac{1}{q} \sum_{j (\bmod\ q)} \Big(\sum_{0 \le |n-x| \le h} \Big(1 - \frac{|n-x|}{h}\Big) e_q(j(n-x)) \Big) e_q(xj) - \frac{h}{q}$$

where $j(\bmod\ q)$ is a sum over a complete set of residues (mod q), and a $*-$sum will restrict to reduced residues; see the following, soon after equation (1), for the coefficients in this FINITE FOURIER EXPANSION.

Remark. We point out that the DIVISOR RANGES of q are fundamental, at least because the sum into $\widetilde{\chi}_q(x,h)$ definition above becomes SPORADIC, *i.e. with at most one summand*, esp. when $h = o(q)$, see Lemma 1 proof. In next §3 we'll distinguish the "low range", of the divisors, from the "sporadic range".

Before going into technical details, we give a sketch of the Theorem Proof.

1)

Our calculations will provide, for a suitable arithmetic function $g_3(q)$, which depends also on N and h,

$$\widetilde{J}_3(N,h) \lll \sum_{x \sim N} \Big| \sum_{q \le x/[N^{1/3}]} g_3(q) \widetilde{\chi}_q(x,h) \Big|^2 + N^{1/3} h^2.$$

This kind of range-reduction for the divisors has been played in [C0], and we'll refer to it as the $k-$FOLDING (see that for the case $k=2$ this amounts to the Dirichlet hyperbola trick, say the flipping of the divisors); we'll do only $3-$folding, but during the Theorem proof.

2)

We need to "MOLLIFY" the modified Selberg integral (a technique applied to moments, for example; here, it will be the easiest mollification possible, namely arithmetic average !), see soon after this sketch.

3)

Then, we will apply the general Proposition, following, for the ranges of the divisors not depending on the long summation variable x.

4)

We will treat the last range of the divisors, depending on x, through a particular argument, which we'll see in next section, during the Theorem proof.

5)

We will build up every range together to finally prove the Theorem.

In order to bound $\widetilde{J}_f(N,h)$, we need the AVERAGED MODIFIED SELBERG INTEGRAL, which is a kind of smoothing of it:

$$\widetilde{J}_f^{av}(N,h) \stackrel{def}{=} \frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)-M_f(x,h)\Big|^2,$$

since this arithmetic mean gives rise to the CESARO WEIGHTS (on the, say, "GLOBAL" sum):

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}P(x)=\sum_{h<|x|<N}\left(1-\frac{|x|}{N}\right)P(x)=\sum_{h<|x|\le N}\left(1-\frac{|x|}{N}\right)P(x)=\sum_{|x|>h}C_N(x)P(x),$$

say, defined as

$$C_R(n) \stackrel{def}{=} \left(1-\frac{|n|}{R}\right)_+ = \max\left(0,1-\frac{|n|}{R}\right)$$

(here we abbreviated with $P(x)$ the square of the inner "LOCAL" error, i.e., over the short interval) and then, for all functions $P(x)\ge 0\ \forall x$ we get, rescaling the "long" variable N (in fact, we use $h=o(N)$ now),

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}P(x)\ge\sum_{h<|x|\le\frac{N}{2}}\Big(1-\frac{|x|}{N}\Big)P(x)\ge\frac{1}{2}\sum_{\frac{N}{4}<x\le\frac{N}{2}}P(x)\ \Rightarrow\ \sum_{x\sim N}P(x)\ll\frac{1}{4N}\sum_{X\le 4N}\sum_{h<|x|<X}P(x)$$

which means, in particular for $P(x):=|\sum_n C_h(n-x)f(n)-M_f(x,h)|^2$, compare the above, the bound

$$\widetilde{J}_f(N,h)\ll\widetilde{J}_f^{av}(4N,h).$$

In the Theorem proof, §3, the same procedure will be given for $P(x)=|\sum_q g(q)\widetilde{\chi}_q(x,h)|^2$, with suitable g.

We'll see an interplay between GLOBAL & LOCAL sums, RULED BY the POSITIVE DEFINITE Cesaro WEIGHTS.

By the way, the Discrete Fourier Transform (abbreviated DFT) of the Cesaro weights is, actually, the exponential sum

$$\widehat{C_R}(\alpha) \stackrel{def}{=} \sum_n C_R(n)e(n\alpha)=\sum_{0\le|n|\le R}\left(1-\frac{|n|}{R}\right)e(n\alpha),\quad \forall R\in\mathbb{N}$$

and this is the well-known FEJÉR KERNEL (non-negative, i.e. C_R are positive definite), summed as, $\forall R\in\mathbb{N}$

$$(1)\quad \widehat{C_R}(\alpha)=\frac{1}{R}\Big|\sum_{n\le R}e(n\alpha)\Big|^2=\frac{\sin^2(\pi\alpha R)}{R\sin^2(\pi\alpha)}\ge 0,\ \forall\alpha\in\mathbb{R}\backslash\mathbb{Z}\quad \text{AND}\quad \widehat{C_R}(0)=1+2\sum_{n\le R}\Big(1-\frac{n}{R}\Big)=R.$$

We define, see (0), the FOURIER COEFFICIENTS $\widetilde{F}_h$ of our "SPORADIC FUNCTION" $\widetilde{\chi}_q(x,h)$ as follows:

$$\widetilde{\chi}_q(x,h)=\frac{1}{q}\sum_{j(\mathrm{mod}\ q)}\widetilde{F}_h\left(\frac{j}{q}\right)e_q(xj),\quad \widetilde{F}_h\left(\frac{j}{q}\right)\stackrel{def}{=}\widehat{C_h}\left(\frac{j}{q}\right)=\frac{\sin^2(\pi jh/q)}{h\sin^2(\pi j/q)},\quad \forall j\not\equiv 0(\mathrm{mod}\ q)$$

where the MEAN-VALUE is zero:

$$\widetilde{F}_h\left(\frac{j}{q}\right)\stackrel{def}{=}0,\quad \forall j\equiv 0(\mathrm{mod}\ q).$$

(We'll assume $q|h$ doesn't happen; however, these divisors $q|h$ contribute to our integrals $\lll N$, negligible.)

Recall $\|\alpha\|$ is the DISTANCE TO INTEGERS, so, from $\|\alpha\|=\min(\{\alpha\},1-\{\alpha\})\le\{\alpha\}$ (the FRACTIONAL PART of α), we have $q\|h/q\|\le q\{h/q\}\le h\ \forall h,q\in\mathbb{N}$. This is applied in previous definition of $\widetilde{F}_h$, right now.

The PARSEVAL IDENTITY, in fact, gives:

$$\frac{1}{q^2}\sum_{j(\mathrm{mod}\ q)}\widetilde{F}_h\Big(\frac{j}{q}\Big)^2 \le \frac{1}{q^2}\sum_{j(\mathrm{mod}\ q)}\widetilde{F}_{q\|h/q\|}\left(\frac{j}{q}\right)^2 = \frac{1}{q}\sum_{j(\mathrm{mod}\ q)}\widetilde{\chi}_q\left(j,q\left\|\frac{h}{q}\right\|\right)^2 \le$$

$$\le \frac{1}{q}\sum_{0\le|j|\le\frac{q}{2}}\left(\sum_{\substack{0\le|n-j|<q\|h/q\|\\ n\equiv 0(\mathrm{mod}\ q)}}\left(1-\frac{|n-j|}{q\|h/q\|}\right)-\left\|\frac{h}{q}\right\|\right)^2 \ll \left\|\frac{h}{q}\right\| + \frac{1}{q}\sum_{0\le|j|\le\frac{q}{2}}\sum_{\substack{0\le|n-j|<q\|h/q\|\\ n\equiv 0(\mathrm{mod}\ q)}} 1 \ll \left\|\frac{h}{q}\right\| + \sum_{j<q\left\|\frac{h}{q}\right\|}\frac{1}{q}$$

(see the Remark soon after eq.(0), on sporadic sums, here using their square is $\ll$ the sum itself), whence

$$\frac{1}{q^2}\sum_{j(\mathrm{mod}\ q)}\widetilde{F}_h\Big(\frac{j}{q}\Big)^2 \ll \left\|\frac{h}{q}\right\|. \tag{2}$$

3. Lemmas, a general Proposition and the Proof of the Theorem.

We have to treat our mean-squares depending on the divisor ranges. The low range will be treated first, from Large-Sieve-inequality type arguments, namely applying the well-spaced property of Farey fractions; then, after this range, we have a kind of sporadic sums over the $n-$variable which can be treated, in mean-square and after averaging on the divisors, through a majorant principle, resembling the Hardy majorant principle for exponential sums mean-squares (proved by Parseval identity); finally, the majorant mean-square will be treated through the Dirichlet hyperbola trick, since it gives a kind of modified Selberg integral for $f = \mathbf{d}$.

We start with a Lemma treating the low divisors. Errors giving $\lll Nh$ will be called DIAGONAL-TYPE.

LEMMA 0 (LOW DIVISORS). *Let $N,h\in\mathbb{N}$ with $h=h(N)\to\infty$, $h=o(\sqrt{N})$ when $N\to\infty$. Assume that $g:\mathbb{N}\to\mathbb{C}$ is essentially bounded, with $g(q)$ depending also at most on N,h. Then*

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{q\le\sqrt{N}} g(q)\widetilde{\chi}_q(x,h)\Big|^2 \lll Nh.$$

PROOF. We expand $\widetilde{\chi}_q(x,h)=\frac{1}{q}\sum_{r(\mathrm{mod}\ q)}\widetilde{F}_h(r/q)e_q(xr)$, then :

$$\widetilde{\chi}_q(x,h)=\frac{1}{q}\sum_{\ell|q}\ \sum_{j<\ell}{}^{*}\ \widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj),$$

where the $*-$summation is over $j(\mathrm{mod}\ \ell)$ coprime to ℓ, simply considering the greatest common divisor (GCD), i.e. $(r,q)=q/\ell$, say, and then setting $j:=r/(r,q)$. This giving rise to Farey fractions $j/\ell\in]0,1[$ (from coprimality $0,1$ are never attained), which we know to be well-spaced, i.e. say

$$\delta \overset{def}{=} \left\|\frac{j}{\ell}-\frac{r}{t}\right\| = \left\|\frac{jt-r\ell}{\ell t}\right\| \ge \frac{1}{\ell t} \ge \frac{1}{N}, \quad \forall \frac{j}{\ell}\neq\frac{r}{t},$$

since $\ell|q\Rightarrow \ell\le q\Rightarrow \ell\le\sqrt{N}$, same for t here. Then, we expand the LHS (left hand side) above, getting

$$\sum_{\ell\le\sqrt{N}}\sum_{d\le\frac{\sqrt{N}}{\ell}}\frac{g(\ell d)}{\ell d}\sum_{t\le\sqrt{N}}\sum_{m\le\frac{\sqrt{N}}{t}}\frac{\overline{g(tm)}}{tm}\ \sum_{j<\ell}{}^{*}\ \widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\ \widetilde{F}_h\left(\frac{r}{t}\right)\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}e(\delta x)$$

which can be distinguished in DIAGONAL ($\delta=0$) and NON-DIAGONAL terms ($\delta>0$); the former are

$$\sum_{\ell\le\sqrt{N}}\Big|\sum_{d\le\frac{\sqrt{N}}{\ell}}\frac{g(\ell d)}{d}\Big|^2\frac{1}{\ell^2}\ \sum_{j<\ell}{}^{*}\ \widetilde{F}_h\Big(\frac{j}{\ell}\Big)^2\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}1 \lll N\sum_{\ell\le\sqrt{N}}\left\|\frac{h}{\ell}\right\| \lll N\Big(\sum_{\ell\le 2h}1+\sum_{2h<\ell\le\sqrt{N}}\frac{h}{\ell}\Big),$$

from (2) above for the mean-square of the Fourier coefficients; this is $\lll Nh$, in fact, DIAGONAL-TYPE. The non-diagonal terms are treated applying the well-spaced property (see [CS], Lemma 2) because

$$\sum_{\ell\le\sqrt{N}}\sum_{d\le\frac{\sqrt{N}}{\ell}}\frac{g(\ell d)}{\ell d}\sum_{t\le\sqrt{N}}\sum_{m\le\frac{\sqrt{N}}{t}}\frac{\overline{g(tm)}}{tm}\ \sum_{\substack{j<\ell\\ \delta>0}}^{*}\sum_{r<t}^{*}\widetilde{F}_h\Big(\frac{j}{\ell}\Big)\widetilde{F}_h\Big(\frac{r}{t}\Big)\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}e(\delta x)\lll$$

$$\lll\sum_{\ell\le\sqrt{N}}\frac{1}{\ell}\sum_{t\le\sqrt{N}}\frac{1}{t}\ \sum_{\substack{j<\ell\\ \delta\ge 1/N}}^{*}\sum_{r<t}^{*}\widetilde{F}_h\Big(\frac{j}{\ell}\Big)\widetilde{F}_h\Big(\frac{r}{t}\Big)\frac{1}{\delta}$$

from $\sum_x e(\delta x)\ll 1/\delta$ [D, chap.25]; that, after Cauchy inequality and renumbering (if necessary) of the distinct Farey fractions (see quoted Lemma proof for details), is, again from (2),

$$\lll N\sum_{\ell\le\sqrt{N}}\frac{1}{\ell^2}\ \sum_{j<\ell}^{*}\widetilde{F}_h\Big(\frac{j}{\ell}\Big)^2\lll N\Big(\sum_{\ell\le 2h}1+\sum_{2h<\ell\le\sqrt{N}}\frac{h}{\ell}\Big)\lll Nh.\ \square$$

We indicate g^{av} instead of g, to refer to the procedure in Theorem Proof, §3. Hereon $\|G\|_\infty\stackrel{def}{=}\max_q|G(q)|$.

We give now the majorant principle for the high divisors (i.e., the SPORADIC RANGE, see (0) Remark).

LEMMA 1 (MAJORANT PRINCIPLE). *Let $N,h\in\mathbb{N}$ with $h=h(N)\to\infty$, $h=o(\sqrt{N})$ as $N\to\infty$. Assume that $g^{av}:\mathbb{N}\to\mathbb{C}$ and $G:\mathbb{N}\to\mathbb{R}$ have support up to $Q=Q(N,h)=o(N)$ with $\sqrt{N}=o(Q)$, and both their values $g^{av}(q),G(q)$ depend also at most on the variables N,h. Assume furthermore $g^{av},G\lll 1$ and*

$$\left|\sum_{n\le\frac{Q}{\ell}}\frac{g^{av}(\ell n)}{n}\right|\ll\left(\sum_{n\le\frac{QL}{\ell}}\frac{G(\ell n)}{n}\right)\quad\forall\ell\le Q.$$

Then

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{q\le Q}g^{av}(q)\widetilde{\chi}_q(x,h)\right|^2\lll\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{q\le QL}G(q)\widetilde{\chi}_q(x,h)\right|^2+h\left|\sum_{q\le Q}g^{av}(q)\right|^2+h^3+$$

$$+\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\sum_{j<\ell}^{*}\widetilde{F}_h\Big(\frac{j}{\ell}\Big)e_\ell(xj)\right|^2+$$

$$+h\left|\sum_{q\le QL}G(q)-\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{d}\right)\sum_{t|\ell}\frac{\mu(t)}{t}\right|^2.$$

In particular, FIX $\varepsilon_0>0$ *arbitrarily* SMALL AND CHOOSE $G(q)=G_0(q)\stackrel{def}{=}K_{\varepsilon_0}(N,Q)-q^{\varepsilon_0}$, *with*

$$K_{\varepsilon_0}(N,Q)\stackrel{def}{=}\frac{\frac{Q^{\varepsilon_0}}{1+\varepsilon_0}\left(\sum_{d<L}\frac{1}{d^{1-\varepsilon_0}}+L^{1+\varepsilon_0}\sum_{d\ge L}\frac{1}{d^2}\right)}{\sum_{d<L}\frac{1}{d}+L\sum_{d\ge L}\frac{1}{d^2}}\qquad(\lll 1),$$

so to get

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{q\le Q}g^{av}(q)\widetilde{\chi}_q(x,h)\Big|^2\lll Nh+\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{q\le QL}G_0(q)\widetilde{\chi}_q(x,h)\Big|^2+h\Big|\sum_{q\le Q}g^{av}(q)\Big|^2+Qh^2.$$

PROOF. We expand the LHS, as above (see Lemma 0) :

$$\sum_{\ell\le Q}\sum_{d\le\frac{Q}{\ell}}\frac{g^{av}(\ell d)}{\ell d}\sum_{t\le Q}\sum_{m\le\frac{Q}{t}}\frac{\overline{g^{av}(tm)}}{tm}\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\,\widetilde{F}_h\left(\frac{r}{t}\right)\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}e(\delta x)$$

where, as in Lemma 0, we abbreviate $\delta=\left\|\frac{j}{\ell}-\frac{r}{t}\right\|$. Since from (1) we have for the DFT of Cesaro weights :

$$\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}e(\alpha x)=\sum_{0\le|x|<N}\left(1-\frac{|x|}{N}\right)e(\alpha x)=\widehat{C_N}(\alpha)\ge 0,\quad\forall\alpha\in\mathbb{R},$$

our term above misses the "short sum"

$$\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|\le h}e(\delta x)=\sum_{0\le|x|\le h}e(\delta x),$$

which, in order to exploit positivity, we reintroduce:

$$\sum_{\ell\le Q}\sum_{d\le\frac{Q}{\ell}}\frac{g^{av}(\ell d)}{\ell d}\sum_{t\le Q}\sum_{m\le\frac{Q}{t}}\frac{\overline{g^{av}(tm)}}{tm}\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\,\widetilde{F}_h\left(\frac{r}{t}\right)\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}e(\delta x)$$

$$-\sum_{\ell\le Q}\sum_{d\le\frac{Q}{\ell}}\frac{g^{av}(\ell d)}{\ell d}\sum_{t\le Q}\sum_{m\le\frac{Q}{t}}\frac{\overline{g^{av}(tm)}}{tm}\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\,\widetilde{F}_h\left(\frac{r}{t}\right)\sum_{0\le|x|\le h}e(\delta x)=$$

$$=\sum_{\ell\le Q}\sum_{d\le\frac{Q}{\ell}}\frac{g^{av}(\ell d)}{\ell d}\sum_{t\le Q}\sum_{m\le\frac{Q}{t}}\frac{\overline{g^{av}(tm)}}{tm}\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\,\widetilde{F}_h\left(\frac{r}{t}\right)\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}e(\delta x)+$$

$$+O\left(\sum_{0\le|x|\le h}\Big|\sum_{q\le Q}g^{av}(q)\widetilde{\chi}_q(x,h)\Big|^2\right)\ll$$

$$\ll\sum_{\ell\le Q}\left|\sum_{d\le\frac{Q}{\ell}}\frac{g^{av}(\ell d)}{\ell d}\right|\sum_{t\le Q}\left|\sum_{m\le\frac{Q}{t}}\frac{g^{av}(tm)}{tm}\right|\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\,\widetilde{F}_h\left(\frac{r}{t}\right)\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}e(\delta x)+$$

$$+\sum_{0\le|x|\le h}\Big|\sum_{q\le Q}g^{av}(q)\widetilde{\chi}_q(x,h)\Big|^2$$

and, thanks to our hypothesis on, say, RAMANUJAN COEFFICIENTS (actually, $\sum_m\frac{g^{av}(\ell m)}{\ell m}$ are $\ell-$th coefficients of the A.F. $g^{av}*\mathbf{1}$, here, compare [C0], esp.)

$$\ll\sum_{\ell\le Q}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\sum_{t\le Q}\left(\sum_{m\le\frac{QL}{t}}\frac{G(tm)}{tm}\right)\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)\ \sum_{r<t}{}^{*}\,\widetilde{F}_h\left(\frac{r}{t}\right)\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}e(\delta x)+R(g^{av}),$$

SAY, where

$$R(g^{av})\stackrel{def}{=}\sum_{0\le|x|\le h}\Big|\sum_{q\le Q}g^{av}(q)\widetilde{\chi}_q(x,h)\Big|^2$$

giving

$$\ll\frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}\left|\sum_{\ell\le Q}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\ \sum_{j<\ell}{}^{*}\,\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right|^2+R(g^{av})\ll$$

$$\ll \frac{1}{N}\sum_{X\le N}\sum_{0\le|x|<X}\left|\sum_{q\le QL}G(q)\widetilde{\chi}_q(x,h)-\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\ {\sum_{j<\ell}}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right|^2+R(g^{av}),$$

going back to $\widetilde{\chi}_q(x,h)$; and, using the Ramanujan sums, following:

$$c_\ell(n)\overset{def}{=}\ {\sum_{j<\ell}}^{*}e_\ell(jn)=\sum_{\substack{d|\ell\\ d|n}}d\mu\left(\frac{\ell}{d}\right)=\varphi(\ell)\frac{\mu(\ell/(\ell,n))}{\varphi(\ell/(\ell,n))}$$

recalling $\varphi(n)\overset{def}{=}|\{m\le n:(m,n)=1\}|$ is the Euler function, we get

$${\sum_{j<\ell}}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)=\mathbf{1}_{\ell>1}\sum_{0\le|n-x|\le h}\left(1-\frac{|n-x|}{h}\right)c_\ell(n)$$

(i.e., it's 0 if $\ell=1$, this above $\forall\ell>1$), obtaining for the previous LHS

$$\ll\sum_{0\le|x|\le h}\left|\sum_{q\le QL}G(q)\widetilde{\chi}_q(x,h)-\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\sum_{0\le|n-x|\le h}\left(1-\frac{|n-x|}{h}\right)c_\ell(n)\right|^2+$$

$$+\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{q\le QL}G(q)\widetilde{\chi}_q(x,h)-\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\ {\sum_{j<\ell}}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right|^2+R(g^{av})\ll$$

$$\ll\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{q\le QL}G(q)\widetilde{\chi}_q(x,h)\right|^2+R(g^{av})+R(G)+$$

$$+\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\ {\sum_{j<\ell}}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right|^2,$$

going back to the averaged modified Selberg integrals, say, with the remainder terms:

$$R(G)\overset{def}{=}\sum_{0\le|x|\le h}\left|\sum_{q\le QL}G(q)\widetilde{\chi}_q(x,h)-\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{\ell d}\right)\sum_{0\le|n-x|\le h}\left(1-\frac{|n-x|}{h}\right)c_\ell(n)\right|^2.$$

See that, from the property

$$n\neq0\ \Rightarrow\ \sum_{q|n}g(q)\lll\|g\|_\infty,$$

we get (using $C_h\ge0$ and $C_h\ll1$)

$$R(g^{av})\lll\sum_{0\le|x|\le h}\Big(\Big|\sum_{\substack{0\le|n-x|\le h\\ n\neq0}}C_h(n-x)\Big|^2+\Big|\sum_{q\le Q}g^{av}(q)C_h(x)\Big|^2+\Big|h\sum_{q\le Q}\frac{g^{av}(q)}{q}\Big|^2\Big)\lll$$

$$\lll h^3+\sum_{0\le|x|\le h}C_h(x)^2\Big|\sum_{q\le Q}g^{av}(q)\Big|^2\lll h^3+h\Big|\sum_{q\le Q}g^{av}(q)\Big|^2$$

(with this last $q-$sum coming from the single term $n=0$), and analogously

$$R(G)\lll h^3+h\Big|\sum_{q\le QL}G(q)-\sum_{Q<\ell\le QL}\Big(\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{d}\Big)\sum_{t|\ell}\frac{\mu(t)}{t}\Big|^2+\sum_{0\le|x|\le h}\Big|\sum_{Q<\ell\le QL}\frac{1}{\ell}\sum_{\substack{0\le|n-x|\le h\\ n\neq0}}|c_\ell(n)|\Big|^2.$$

Here we used $c_\ell(0)=\varphi(\ell)$, see above, and see [Te] for the well-known $\varphi(\ell)/\ell=\sum_{t|\ell}\mu(t)/t$; then

$$|c_\ell(n)|\le \frac{\varphi(\ell)}{\varphi(\ell/(\ell,n))}\le \varphi((\ell,n))\le (\ell,n)$$

from the above formulæ for the Ramanujan sums (with (ℓ,n) the GCD of ℓ,n) gives

$$\sum_{0\le|x|\le h}\Big|\sum_{Q<\ell\le QL}\frac{1}{\ell}\sum_{\substack{0\le|n-x|\le h\\ n\neq 0}}|c_\ell(n)|\Big|^2 \lll \sum_{0\le|x|\le h}\Big(\sum_{Q<\ell\le QL}\sum_{t|\ell}\frac{t}{\ell}\sum_{\substack{\frac{x-h}{t}\le m\le\frac{x+h}{t}\\ m\neq 0}}1\Big)^2$$

$$\lll \sum_{0\le|x|\le h}\Big(\sum_{t\le QL}\sum_{\substack{\frac{x-h}{t}\le m\le\frac{x+h}{t}\\ m\neq 0}}1\Big)^2 \lll \sum_{0\le|x|\le h}\Big(\sum_{\substack{x-h\le n\le x+h\\ n\neq 0}}\sum_{\substack{t\le QL\\ t|n}}1\Big)^2 \lll \sum_{0\le|x|\le h}\Big(\sum_{\substack{x-h\le n\le x+h\\ n\neq 0}}\mathbf{d}(|n|)\Big)^2 \lll h^3.$$

Our bound with general G is proved.

We build up, now, an explicit $G=G_0$ in order to get our second bound. We start from the simple fact that, in order to get the Ramanujan coefficients upper bound for g^{av} we need, since $g^{av}\lll 1$, to consider at the outset an initial smooth function, say $G(q)=q^\varepsilon$, with $\varepsilon>0$ arbitrarily small (after ε_0), and we calculate:

$$\sum_{q\le QL}G(q)-\sum_{Q<\ell\le QL}\sum_{d\le\frac{QL}{\ell}}\frac{G(\ell d)}{d}\sum_{t|\ell}\frac{\mu(t)}{t}=S_1(G)-S_2(G),$$

say, with

$$S_1(q^\varepsilon)\stackrel{def}{=}\sum_{q\le QL}q^\varepsilon\sim\frac{1}{1+\varepsilon}(QL)^{1+\varepsilon}$$

by partial summation [Te] and writing $\sim$ to neglect terms $\lll 1$, and, say,

$$S_2(q^\varepsilon)\stackrel{def}{=}\sum_{Q<\ell\le QL}\sum_{d\le\frac{QL}{\ell}}\frac{(\ell d)^\varepsilon}{d}\sum_{t|\ell}\frac{\mu(t)}{t}=\sum_{t\le QL}\frac{\mu(t)}{t^{1-\varepsilon}}\sum_{d<L}\frac{1}{d^{1-\varepsilon}}\sum_{\frac{Q}{t}<m\le\frac{QL}{td}}m^\varepsilon\sim$$

$$\sim\frac{1}{1+\varepsilon}\sum_{t\le QL}\frac{\mu(t)}{t^{1-\varepsilon}}\sum_{d<L}\frac{1}{d^{1-\varepsilon}}\left(\left(\frac{QL}{td}\right)^{1+\varepsilon}-\left(\frac{Q}{t}\right)^{1+\varepsilon}\right)\sim\frac{Q^{1+\varepsilon}}{1+\varepsilon}\left(\sum_{t\le QL}\frac{\mu(t)}{t^2}\right)\left(L^{1+\varepsilon}\sum_{d<L}\frac{1}{d^2}-\sum_{d<L}\frac{1}{d^{1-\varepsilon}}\right)\sim$$

$$\sim\frac{Q^{1+\varepsilon}}{1+\varepsilon}\frac{1}{\zeta(2)}\left(L^{1+\varepsilon}\sum_{d<L}\frac{1}{d^2}-\sum_{d<L}\frac{1}{d^{1-\varepsilon}}\right),$$

from the well-known $\sum_{n\le x}\frac{\mu(n)}{n}=\frac{1}{\zeta(2)}+O(1/x)$, as $x\to\infty$, whence

$$S_1(q^\varepsilon)-S_2(q^\varepsilon)\sim Q\left(\frac{Q^\varepsilon}{\zeta(2)(1+\varepsilon)}\sum_{d<L}\frac{1}{d^{1-\varepsilon}}+\frac{Q^\varepsilon L^{1+\varepsilon}}{(1+\varepsilon)}\left(1-\frac{1}{\zeta(2)}\sum_{d<L}\frac{1}{d^2}\right)\right)\sim$$

$$\sim\frac{Q}{\zeta(2)}\left(\frac{Q^\varepsilon}{1+\varepsilon}\sum_{d<L}\frac{1}{d^{1-\varepsilon}}+\frac{Q^\varepsilon L^{1+\varepsilon}}{1+\varepsilon}\sum_{d\ge L}\frac{1}{d^2}\right),$$

using $\sum_{d<L}\frac{1}{d^2}=\zeta(2)-\sum_{d\ge L}\frac{1}{d^2}$. More easily, the same calculations give

$$S_1(K)-S_2(K)=K\left(QL-\sum_{t\le QL}\frac{\mu(t)}{t}\sum_{d<L}\frac{1}{d}\sum_{\frac{Q}{t}<m\le\frac{QL}{td}}1\right)\sim KQ\left(L-\sum_{t\le QL}\frac{\mu(t)}{t^2}\sum_{d<L}\frac{1}{d}\left(\frac{L}{d}-1\right)\right)\sim$$

$$\sim K\frac{Q}{\zeta(2)}\left(\sum_{d<L}\frac{1}{d}+L\sum_{d\ge L}\frac{1}{d^2}\right)$$

whence the $G_0(q)$ choice above with $K=K_{\varepsilon_0}(N,Q)$ is due to $n=0$ term for $G=G_0$, bounded as $\lll 1$, from

$$S_1(G_0)-S_2(G_0)=(S_1(K)-S_2(K))-(S_1(q^{\varepsilon_0})-S_2(q^{\varepsilon_0}))\lll 1.$$

After having bounded some G_0 error terms, we check that the hypothesis on the Ramanujan coefficients with $G=G_0$ holds:

$$\left|\sum_{n\le\frac{Q}{\ell}}\frac{g^{av}(\ell n)}{n}\right|\ll\left(\sum_{n\le\frac{QL}{\ell}}\frac{G_0(\ell n)}{n}\right)\quad\forall\ell\le Q$$

and we do it for the general $g^{av}\lll 1$, namely we simply prove:

$$\sum_{n\le\frac{QL}{\ell}}\frac{G_0(\ell n)}{n}\gg_{\varepsilon_0}Q^{\varepsilon_0},$$

actually from the initial asymptotic we get a lower bound, $\forall\ell\le Q$, namely

$$\sum_{n\le\frac{QL}{\ell}}\frac{G_0(\ell n)}{n}=K\left(\log\frac{QL}{\ell}+\gamma+O\left(\frac{\ell}{QL}\right)\right)-\ell^{\varepsilon_0}\sum_{n\le\frac{QL}{\ell}}\frac{1}{n^{1-\varepsilon_0}}\sim K\log\frac{QL}{\ell}+\gamma K-\frac{(QL)^{\varepsilon_0}}{\varepsilon_0}\ge\frac{\gamma(QL)^{\varepsilon_0}}{\varepsilon_0\log L}$$

using now $\sim$ in the classical asymptotic (ratio $\to 1$) meaning, coming from the other (classic) asymptotic :

$$K=K_{\varepsilon_0}(N,Q)\sim\frac{Q^{\varepsilon_0}}{1+\varepsilon_0}\left(\sum_{d<L}\frac{1}{d^{1-\varepsilon_0}}+L^{1+\varepsilon_0}\sum_{d\ge L}\frac{1}{d^2}\right)\frac{1}{\log L}\sim\frac{Q^{\varepsilon_0}L^{\varepsilon_0}}{1+\varepsilon_0}\left(\frac{1}{\varepsilon_0}+1\right)\frac{1}{\log L}\sim\frac{Q^{\varepsilon_0}L^{\varepsilon_0}}{\varepsilon_0}\frac{1}{\log L}.$$

We are left with the estimate of, SAY,

$$\Sigma(G_0)\stackrel{def}{=}\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{G_0(\ell d)}{\ell d}\right)\sum_{j<\ell}{}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right|^2=$$

$$=\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\left|\sum_{Q<\ell\le QL}\left(\sum_{d\le\frac{QL}{\ell}}\frac{K}{d}-\sum_{d\le\frac{QL}{\ell}}\frac{\ell^\varepsilon d^\varepsilon}{d}\right)\frac{1}{\ell}\sum_{j<\ell}{}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right|^2.$$

We start from the relation coming from $\widetilde{\chi}_q(x,h)$ expansion above, due to Möbius inversion (see [T]),

$$\widetilde{\chi}_q(x,h)=\sum_{\ell|q}\left(\frac{1}{\ell}\sum_{j<\ell}{}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)\right)\frac{1}{q/\ell}\quad\Rightarrow\quad\frac{1}{\ell}\sum_{j<\ell}{}^{*}\widetilde{F}_h\left(\frac{j}{\ell}\right)e_\ell(xj)=\sum_{t|\ell}\frac{\mu(\ell/t)}{\ell/t}\widetilde{\chi}_t(x,h),$$

recalling that the Dirichlet convolution inverse of, say, the A.F. $\frac{1}{N}(n)\stackrel{def}{=}\frac{1}{n}$ is, say, the A.F. $\frac{\mu}{N}(n)\stackrel{def}{=}\frac{\mu(n)}{n}$ (since the a.f. $\frac{1}{N}$ is completely multiplicative) and we apply it here, to get as inner $\ell-$sum, inside $\Sigma(G_0)$,

$$\sum_{t\le QL}\widetilde{\chi}_t(x,h)\sum_{\frac{Q}{t}<m\le\frac{QL}{t}}\frac{\mu(m)}{m}\Big(\sum_{d\le\frac{QL}{tm}}\frac{K}{d}-m^\varepsilon t^\varepsilon\sum_{d\le\frac{QL}{tm}}d^{\varepsilon-1}\Big)\lll\sum_{t\le QL}|\widetilde{\chi}_t(x,h)|,$$

trivially, which is clearly

$$\lll \sum_{t\le QL}\sum_{\substack{0\le|n-x|\le h\\ n\equiv 0(t)}} 1+\sum_{t\le QL}\frac{h}{t}\lll \sum_{0\le|n-x|\le h}\sum_{\substack{t\le QL\\ t|n}} 1+h\lll h,$$

since $|x|>h$ implies $n\neq 0$ in this last sum; then, from this trivial bound and the property of being $x-$even

$$\Sigma(G_0)\lll \frac{1}{N}\sum_{X\le N}\sum_{QL^2<x<X}\Big|\sum_{t\le QL}\widetilde{\chi}_t(x,h)\sum_{\frac{Q}{t}<m\le\frac{QL}{t}}\frac{\mu(m)}{m}\Big(\sum_{d\le\frac{QL}{tm}}\frac{K}{d}-m^{\varepsilon}t^{\varepsilon}\sum_{d\le\frac{QL}{tm}}d^{\varepsilon-1}\Big)\Big|^2+Qh^2\lll Qh^2+$$

$$+\frac{1}{N}\sum_{X\le N}\sum_{QL^2<x<X}\Big|\sum_{m\le QL}\frac{\mu(m)}{m}\sum_{\substack{d\le\frac{QL}{m}\\ d<L}}\frac{K}{d}\sum_{\frac{Q}{m}<t\le\frac{QL}{md}}\widetilde{\chi}_t(x,h)-\sum_{m\le QL}\frac{\mu(m)}{m}m^{\varepsilon}\sum_{\substack{d\le\frac{QL}{m}\\ d<L}}\frac{d^{\varepsilon}}{d}\sum_{\frac{Q}{m}<t\le\frac{QL}{md}}t^{\varepsilon}\widetilde{\chi}_t(x,h)\Big|^2,$$

after exchanging sums again. Here, choosing $A\to\infty$, $B=o(x)$ and $h=o(x)$ gives, compare next Lemma,

$$G\lll 1, G'(t)\lll\frac{1}{t}\Rightarrow\sum_{A<t\le B}G(t)\widetilde{\chi}_t(x,h)=\sum_{\frac{x}{B}<q\le\frac{x}{A}}G\left(\frac{x}{q}\right)\widetilde{\chi}_q(x,h)+O_{\varepsilon}\left(N^{\varepsilon}h\left(\frac{h}{x}+\frac{B}{x}+\frac{1}{A}\right)+N^{\varepsilon}\right).$$

We apply this to the ranges of t. In fact, choose $\sqrt{N}\ll Q_0\ll\sqrt{N}$ and, leaving negligible $\lll Qh^2$, split as

$$\Sigma(G_0)\lll\frac{1}{N}\sum_{X\le N}\sum_{QL^2<x<X}\left(\sum_{m\le\frac{Q}{Q_0}}\frac{1}{m}\sum_{\substack{d\le\frac{QL}{m}\\ d<L}}\frac{1}{d}\left|\sum_{\frac{Q}{m}<t\le\frac{QL}{md}}\widetilde{\chi}_t(x,h)\right|^2+\sum_{m\le\frac{Q}{Q_0}}\frac{1}{m}\sum_{\substack{d\le\frac{QL}{m}\\ d<L}}\frac{1}{d}\left|\sum_{\frac{Q}{m}<t\le\frac{QL}{md}}t^{\varepsilon}\widetilde{\chi}_t(x,h)\right|^2\right.$$

$$\left.+\sum_{\frac{Q}{Q_0}<m\le QL}\frac{1}{m}\sum_{\substack{d\le\frac{QL}{m}\\ d<L}}\frac{1}{d}\left|\sum_{\frac{Q}{m}<t\le\frac{QL}{md}}\widetilde{\chi}_t(x,h)\right|^2+\sum_{\frac{Q}{Q_0}<m\le QL}\frac{1}{m}\sum_{\substack{d\le\frac{QL}{m}\\ d<L}}\frac{1}{d}\left|\sum_{\frac{Q}{m}<t\le\frac{QL}{md}}t^{\varepsilon}\widetilde{\chi}_t(x,h)\right|^2\right),$$

applying twice the Cauchy inequality; then, these last two mean-squares are treated, since $m>Q/Q_0\Rightarrow t\ll Q_0L\lll\sqrt{N}$, through Lemma 0, to get $\lll Nh$. For the range $m\le Q/Q_0$ we apply previous trick with $G(t)=1$ and $G(t)=t^{\varepsilon}$, to get again $\lll Nh$ from Lemma 0 and the final bound $\lll Nh+Qh^2$ for $\Sigma(G_0)$. □

We recall that $f=g*\mathbf{1}$ ($g-$support up to $o(N)$ here) gives, for the mean-value arithmetic form, see above,

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{0\le|n-x|\le h}\Big(1-\frac{|n-x|}{h}\Big)f(n)-h\sum_q\frac{g(q)}{q}\Big|^2=\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_q g(q)\widetilde{\chi}_q(x,h)\Big|^2$$

Remark. As above, $g\lll 1\Rightarrow$ these terms inside the square, here above, are both $\lll h$, uniformly w.r.t. x.

We treat the "divisor-like functions", in short intervals, as follows applying the Dirichlet hyperbola trick.

LEMMA 2 (DIRICHLET HYPERBOLA). *Let $N,Q,h\in\mathbb{N}$ with $h\to\infty$ and $h=o(\sqrt{N})$. Assume $Q=Q(N,h)$, i.e. depends at most on N,h and $Q=o(N)$, with $\sqrt{N}=o(Q)$ when $N\to\infty$. Let $G:[\sqrt{N},Q]\to\mathbb{R}$ be essentially bounded and with continuous derivative, satisfying in all this interval $\left|\frac{d}{dt}G(t)\right|\lll\frac{1}{t}$. Furthermore, assume that $G(t)$ may depend, also, at most on the variables N,Q,h. Then*

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{\sqrt{N}<q\le Q}G(q)\widetilde{\chi}_q(x,h)\Big|^2\lll Nh+Qh^2.$$

PROOF. Recalling $\widetilde{\chi}_q(x,h)=\sum_{n\equiv 0(q)} C_h(n-x)-\frac{h}{q}$, our $G*\mathbf{1}$ resembles the divisor function $\mathbf{d}=\mathbf{1}*\mathbf{1}$:

$$\sum_{\sqrt{N}<q\le Q} G(q)\widetilde{\chi}_q(x,h)=\sum_{0\le|n-x|\le h} C_h(n-x)\sum_{q|n,\sqrt{N}<q\le Q} G(q)-h\sum_{\sqrt{N}<q\le Q}\frac{G(q)}{q},$$

whence we recall, also, $x-$summands are EVEN. Before to apply the Dirichlet hyperbola trick, we need to cut the range $x\le QL$, negligible, so to ensure $Q/x\to\infty$, in the following (see the Remark above) :

$$\frac{1}{N}\sum_{X\le N}\sum_{h<|x|\le X}\Big|\sum_{\sqrt{N}<q\le Q} G(q)\widetilde{\chi}_q(x,h)\Big|^2 \lll Qh^2+\frac{1}{N}\sum_{X\le N}\sum_{QL<x<X}\Big|\sum_{\sqrt{N}<q\le Q} G(q)\widetilde{\chi}_q(x,h)\Big|^2$$

and, now, the assumption $x>QL$ allows us to apply meaningfully the Dirichlet trick. In fact,

$$\sum_{0\le|n-x|\le h} C_h(n-x)\sum_{q|n,\sqrt{N}<q\le Q} G(q)=\sum_{\frac{x-h}{Q}\le d<\frac{x+h}{\sqrt{N}}}\ \sum_{\substack{\sqrt{N}<q\le Q\\ \frac{x-h}{d}\le q\le\frac{x+h}{d}}}\left(G\left(\frac{x}{d}\right)+O_{\varepsilon}\left(N^{\varepsilon}\frac{h}{x}\right)\right)C_h(dq-x)=$$

$$=\sum_{\frac{x}{Q}<d\le\frac{x}{\sqrt{N}}} G\left(\frac{x}{d}\right)\sum_q C_h(dq-x)+O_{\varepsilon}\left(N^{\varepsilon}\left(\left(\sum_{\frac{x-h}{Q}\le d\le\frac{x+h}{Q}}+\sum_{\frac{x-h}{\sqrt{N}}\le d\le\frac{x+h}{\sqrt{N}}}\right)\left(\frac{h}{d}+1\right)+\frac{h^2}{x}\right)\right)=$$

$$=\sum_{0\le|n-x|\le h} C_h(n-x)\sum_{d|n,\frac{x}{Q}<d\le\frac{x}{\sqrt{N}}} G\left(\frac{x}{d}\right)+O_{\varepsilon}\left(N^{\varepsilon}\left(h\left(\log\left(\frac{x+h}{x-h}\right)+\frac{Q}{x}\right)+\frac{h}{Q}+\frac{h}{\sqrt{N}}+1+\frac{h^2}{x}\right)\right),$$

using h hypotheses and $\sum_{d\le D}\frac{1}{d}=\log D+\gamma+O(1/D)$, $D\to\infty$, see [Te] and above §2 for γ; whence, due to our hypotheses on h,Q when $N\to\infty$, (compare [C] for the same trick in fixed ranges of divisors)

$$\sum_{0\le|n-x|\le h} C_h(n-x)\sum_{q|n,\sqrt{N}<q\le Q} G(q)=\sum_{0\le|n-x|\le h} C_h(n-x)\sum_{d|n,\frac{x}{Q}<d\le\frac{x}{\sqrt{N}}} G\left(\frac{x}{d}\right)+O_{\varepsilon}\left(N^{\varepsilon}\left(\frac{hQ}{x}+1\right)\right),$$

together with quoted formula for sums of reciprocals, entailing (use $\sqrt{N}=o(Q)$ here, with $x>QL$, too)

$$\sum_{\sqrt{N}<q\le Q}\frac{G(q)}{q}=G(Q)\log\frac{Q}{\sqrt{N}}-\int_{\sqrt{N}}^{Q}\log\frac{t}{\sqrt{N}}G'(t)dt+O_{\varepsilon}\left(\frac{N^{\varepsilon}}{\sqrt{N}}\right)=\int_{\sqrt{N}}^{Q}\frac{G(t)}{t}dt+O_{\varepsilon}\left(\frac{N^{\varepsilon}}{\sqrt{N}}\right)$$

and

$$\sum_{\frac{x}{Q}<m\le\frac{x}{\sqrt{N}}}\frac{G(x/m)}{m}=G(\sqrt{N})\sum_{\frac{x}{Q}<m\le\frac{x}{\sqrt{N}}}\frac{1}{m}-\int_{\frac{x}{Q}}^{\frac{x}{\sqrt{N}}}\sum_{\frac{x}{Q}<m\le t}\frac{1}{m}\frac{d}{dt}G(x/t)dt=$$

$$=G(\sqrt{N})\log\frac{Q}{\sqrt{N}}-\int_{\frac{x}{Q}}^{\frac{x}{\sqrt{N}}}\left(\log t-\log\frac{x}{Q}\right)\frac{d}{dt}G(x/t)dt+O_{\varepsilon}\left(\frac{N^{\varepsilon}Q}{x}\right)=$$

$$=\int_{\frac{x}{Q}}^{\frac{x}{\sqrt{N}}}\frac{G(x/t)}{t}dt+O_{\varepsilon}\left(\frac{N^{\varepsilon}Q}{x}\right)=-\int_{Q}^{\sqrt{N}}\frac{G(u)}{u}du+O_{\varepsilon}\left(\frac{N^{\varepsilon}Q}{x}\right)=\int_{\sqrt{N}}^{Q}\frac{G(t)}{t}dt+O_{\varepsilon}\left(\frac{N^{\varepsilon}Q}{x}\right),$$

from partial summation [Te], after a change of variables, whence (same works for general A,B in previous Lemma 1 proof)

$$h\sum_{\sqrt{N}<q\le Q}\frac{G(q)}{q}=h\sum_{\frac{x}{Q}<d\le\frac{x}{\sqrt{N}}}\frac{G(x/d)}{d}+O_{\varepsilon}\left(\frac{N^{\varepsilon}hQ}{x}\right);$$

that gives

$$\frac{1}{N}\sum_{X\le N}\sum_{QL<x<X}\Big|\sum_{\sqrt{N}<q\le Q}G(q)\widetilde{\chi}_q(x,h)\Big|^2=\frac{1}{N}\sum_{X\le N}\sum_{QL<x<X}\Big|\sum_{\frac{x}{Q}<d\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{d}\Big)\widetilde{\chi}_d(x,h)+O_\varepsilon\Big(N^\varepsilon\Big(\frac{hQ}{x}+1\Big)\Big)\Big|^2$$

$$\lll\frac{1}{N}\sum_{QL<X\le N}\sum_{QL<x<X}\Big|\sum_{\frac{x}{Q}<d\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{d}\Big)\widetilde{\chi}_d(x,h)\Big|^2+\frac{1}{N}\sum_{QL<X\le N}\sum_{QL<x<X}\Big(\frac{h^2Q^2}{x^2}+1\Big)\lll$$

$$\lll\frac{1}{N}\sum_{QL<X\le N}\sum_{QL<x\le X}\Big|\sum_{\frac{x}{Q}<q\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{q}\Big)\widetilde{\chi}_q(x,h)\Big|^2+Qh^2+N,$$

whence we need only to show that

$$\frac{1}{N}\sum_{QL<X\le N}\sum_{QL<x\le X}\Big|\sum_{\frac{x}{Q}<q\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{q}\Big)\widetilde{\chi}_q(x,h)\Big|^2\lll Nh.$$

This is accomplished easily, since the Dirichlet hyperbola has now given us the divisor ranges below the square-root of N, i.e. we may apply the Large Sieve arguments; however, the $q-$ranges depend upon x like the G function, whence we can not apply the classic (esp., see [B]) Large Sieve inequality. However, we apply a dyadic argument, namely

$$\frac{1}{N}\sum_{QL<X\le N}\sum_{QL<x\le X}\Big|\sum_{\frac{x}{Q}<q\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{q}\Big)\widetilde{\chi}_q(x,h)\Big|^2\lll\frac{1}{N}\sum_{QL<X\le N}\max_{QL\ll M\ll X}\sum_{x\sim M}\Big|\sum_{\frac{x}{Q}<q\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{q}\Big)\widetilde{\chi}_q(x,h)\Big|^2$$

then we follow Lemma 0 line of proof, using this time

$$\frac{d}{dt}\left(G\Big(\frac{t}{q_1}\Big)G\Big(\frac{t}{q_2}\Big)\right)=\frac{1}{q_1}G'\Big(\frac{t}{q_1}\Big)G\Big(\frac{t}{q_2}\Big)+\frac{1}{q_2}G'\Big(\frac{t}{q_2}\Big)G\Big(\frac{t}{q_1}\Big)\lll\frac{1}{t}$$

and, after partial summation, where the $x-$range may depend upon moduli q_1,q_2 now, together with [D,ch.25] (in Lemma 0 proof the $x-$sum was estimated only using [D,ch.25] there)

$$\sum_x G\Big(\frac{x}{q_1}\Big)G\Big(\frac{x}{q_2}\Big)e(\delta x)\lll\frac{1}{\delta},$$

we apply Lemma 3 of [CS] in order to get (all logarithms we loose in the quoted Lemma are now inside $\lll$, here), together with (2) (similar to the same mean-square for the Fourier coefficients in [CS] for $\chi_q(x)$, another "sporadic function")

$$\sum_{x\sim M}\Big|\sum_{\frac{x}{Q}<q\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{q}\Big)\widetilde{\chi}_q(x,h)\Big|^2\lll\Big(M+\Big(\frac{M}{\sqrt{N}}\Big)^2\Big)h\lll\Big(M+\frac{M^2}{N}\Big)h,$$

whence we finally get the required

$$\frac{1}{N}\sum_{QL<X\le N}\sum_{QL<x\le X}\Big|\sum_{\frac{x}{Q}<q\le\frac{x}{\sqrt{N}}}G\Big(\frac{x}{q}\Big)\widetilde{\chi}_q(x,h)\Big|^2\lll\frac{1}{N}\sum_{X\le N}\Big(X+\frac{X^2}{N}\Big)h\lll Nh.\ \square$$

The following Proposition is a kind of "Fundamental Lemma", for the averaged modified Selberg integral.
Proposition. *Let* $N,h\in\mathbb{N}$ *with* $h=h(N)\to\infty$, $h=o(\sqrt{N})$ *when* $N\to\infty$. *Assume that* $g:\mathbb{N}\to\mathbb{C}$ *has support up to* $Q=Q(N,h)=o(N)$ *and* $g(q)$ *also depends at most on the variables* N,h. *Then*

$$g\lll 1\ \Rightarrow\ \frac{1}{N}\sum_{X\le N}\sum_{h<|x|<X}\Big|\sum_{q\le Q}g(q)\widetilde{\chi}_q(x,h)\Big|^2\lll Nh+\Big|\sum_{q\le Q}g(q)\Big|^2h+Qh^2.$$

PROOF. We simply gather the three Lemmas, with $G = G_0$ in Lemma 2, to get the bound easily. $\square$

We are ready to collect all the properties of $\widetilde{J}_3(N,h)$, together with our Lemmas and the Proposition, to prove our Theorem.

PROOF OF THE THEOREM. We start defining, here $N < x \le 2N$, recalling $C_h(n-x) = \max(0, 1-|n-x|/h)$,

$$\widetilde{S}_k(x,h) \stackrel{def}{=} \sum_{0\le|n-x|\le h} C_h(n-x)d_k(n) = \sum_{\substack{n_1 \\ x-h\le n_1\cdots n_k\le x+h}}\cdots\sum_{n_k} C_h(n_1\cdots n_k - x)$$

for which we want to apply $k-$FOLDING (compare [C0,§4]) and, in particular, we treat the case $k=3$, to get

$$\widetilde{S}_3(x,h) \stackrel{def}{=} \sum_{0\le|abc-x|\le h} C_h(abc-x) = \sum_{\substack{a\ge(N-h)^{1/3} \\ x-h\le abc\le x+h}}\sum_b\sum_c C_h(abc-x) + \sum_{\substack{a<(N-h)^{1/3} \\ x-h\le abc\le x+h}}\sum_b\sum_c C_h(abc-x) =$$

$$= \sum_{\substack{b \\ bc=q\le\frac{x+h}{(N-h)^{1/3}}}}\sum_c \sum_{\substack{a\ge(N-h)^{1/3} \\ \frac{x-h}{q}\le a\le\frac{x+h}{q}}} C_h(aq-x) + \sum_{\substack{b \\ bc=q}}\sum_c \sum_{\substack{a<(N-h)^{1/3} \\ \frac{x-h}{q}\le a\le\frac{x+h}{q}}} C_h(aq-x)$$

but, since $C_h \ll 1$ (uniformly, say $0\le|C_h|\le 1$), abbreviating $M := [(N-h)^{1/3}]$,

$$\sum_{\substack{b \\ bc=q\le\frac{x+h}{M}}}\sum_c \sum_{\substack{a\ge M \\ \frac{x-h}{q}\le a\le\frac{x+h}{q}}} C_h(aq-x) = \sum_{q\le\frac{x-h}{M}} \mathbf{d}(q) \sum_{\frac{x-h}{q}\le a\le\frac{x+h}{q}} C_h(aq-x) + O\left(\sum_{\frac{x-h}{M}\le q\le\frac{x+h}{M}} \mathbf{d}(q) \sum_{\frac{x-h}{q}\le a\le\frac{x+h}{q}} 1\right) =$$

$$= \sum_{q\le\frac{x}{M}} \mathbf{d}(q) \sum_{\frac{x-h}{q}\le a\le\frac{x+h}{q}} C_h(aq-x) + O\left(\sum_{\frac{x-h}{M}\le q\le\frac{x+h}{M}} \mathbf{d}(q) \sum_{\frac{x-h}{q}\le a\le\frac{x+h}{q}} 1\right),$$

(see that M instead of $(N-h)^{1/3}$ gives a negligible error) with this remainder (recall $h = o(\sqrt{N})$, here)

$$\ll \sum_{\frac{x-h}{M}\le q\le\frac{x+h}{M}} \mathbf{d}(q)\left(\frac{h}{q}+1\right) \lll \sum_{\frac{x-h}{M}\le q\le\frac{x+h}{M}} \frac{h}{q} + \left(\frac{h}{M}+1\right) \lll \left(\frac{h}{N^{2/3}}+1\right)\left(\frac{h}{M}+1\right) \lll \left(\frac{h}{N^{1/3}}+1\right),$$

(we'll apply these bounds also for the $\mathbf{d}^{(1)}(q) \lll 1$ and $\mathbf{d}^{(2)}(q) \lll 1$, following) whence

$$\widetilde{S}_3(x,h) \sim \sum_{q\le\frac{x}{M}} \mathbf{d}(q) \sum_a C_h(aq-x) + \sum_{\substack{b \\ bc=q}}\sum_c \sum_{\substack{a<M \\ \frac{x-h}{q}\le a\le\frac{x+h}{q}}} C_h(aq-x),$$

where, as usual, we mean with $\sim$ that we have negligible (i.e., within final remainders) error terms. Switch

$$\sum_{\substack{b \\ bc=q}}\sum_c \sum_{\substack{a<M \\ \frac{x-h}{q}\le a\le\frac{x+h}{q}}} C_h(aq-x) = \sum_{\substack{a<M \\ ab=q}}\sum_b \sum_{\frac{x-h}{q}\le m\le\frac{x+h}{q}} C_h(mq-x) =$$

$$= \sum_{\substack{a<M \\ ac=q\le\frac{x+h}{M}}}\sum_c \sum_{\substack{b\ge M \\ \frac{x-h}{q}\le b\le\frac{x+h}{q}}} C_h(bq-x) + \sum_{\substack{a<M \\ ab=q}}\sum_{b<M} \sum_{\frac{x-h}{q}\le m\le\frac{x+h}{q}} C_h(mq-x) \sim$$

$$\sim \sum_{q\le\frac{x}{M}} \mathbf{d}^{(1)}(q)\sum_b C_h(bq-x) + \sum_{q\le\frac{x}{M}} \mathbf{d}^{(2)}(q)\sum_m C_h(mq-x) \;\Rightarrow\; \widetilde{S}_3(x,h) \sim \sum_{q\le\frac{x}{M}} g_3(q)\sum_a C_h(aq-x),$$

because, from $mq \ge x-h > N-h$, we can't have $m < M$, so $m \ge M$ is "for free", with, say :

$$g_3 \overset{def}{=} \mathbf{d}^{(0)} + \mathbf{d}^{(1)} + \mathbf{d}^{(2)}, \quad \mathbf{d}^{(0)}(q) \overset{def}{=} \mathbf{d}(q), \ \mathbf{d}^{(1)}(q) \overset{def}{=} \sum_{\substack{a<M \\ ab=q}} \sum_{b} 1 = \sum_{a|q, a<M} 1, \ \mathbf{d}^{(2)}(q) \overset{def}{=} \sum_{\substack{a<M \\ ab=q}} \sum_{b<M} 1.$$

We give the mean-value arithmetic form $\widetilde{M}_3(x,h)$ (with negligible switching from M to $N^{1/3}$) defined as :

$$\frac{\widetilde{M}_3(x,h)}{h} \overset{def}{=} \sum_{q \le \frac{x}{N^{1/3}}} \frac{g_3(q)}{q} = \sum_{q \le \frac{x}{N^{1/3}}} \frac{\mathbf{d}(q)}{q} + \sum_{d_1 \le N^{1/3}} \frac{1}{d_1} \sum_{d_2 \le \frac{x}{d_1 N^{1/3}}} \frac{1}{d_2} + \sum_{d_1 \le N^{1/3}} \frac{1}{d_1} \sum_{d_2 \le \min\left(N^{1/3}, \frac{x}{d_1 N^{1/3}}\right)} \frac{1}{d_2}.$$

However, since $x > N$, this $\min(N^{1/3}, \frac{x}{d_1 N^{1/3}}) = N^{1/3}$ and our third sum becomes a square:

$$\frac{\widetilde{M}_3(x,h)}{h} = \sum_{q \le \frac{x}{N^{1/3}}} \frac{\mathbf{d}(q)}{q} + \sum_{d_1 \le N^{1/3}} \frac{1}{d_1} \sum_{d_2 \le \frac{x}{d_1 N^{1/3}}} \frac{1}{d_2} + \Big(\sum_{d \le N^{1/3}} \frac{1}{d} \Big)^2.$$

We wish to prove that arithmetic and analytic form are close to one another, namely:

$$(*) \qquad \widetilde{M}_3(x,h) - M_3(x,h) \lll \frac{h}{N^{1/3}}$$

In order to do this, we need Amitsur's Formula with Tull's Error Term, namely

$$\sum_{q \le Q} \frac{\mathbf{d}(q)}{q} = \frac{\log^2 Q}{2} + 2\gamma \log Q + (\gamma^2 + 2\gamma_1) + O\Big(\frac{1}{\sqrt{Q}}\Big).$$

(Amitsur, an algebraist, derived a symbolic method [A] to calculate mean-terms of asymptotic formulæ, while Tull, a student of Bateman, gave a refined partial summation, see Lemma [Tu], to transfer error terms from, say, here, the formula for the $\mathbf{d}(q)$ sum, esp. Dirichlet's classical $O(\sqrt{Q})$, to this for the $\mathbf{d}(q)/q$ sum.).

Here, γ & γ_1 are defined in the introduction (of course, $Q \to \infty$) and we need, of course, also ($D \to \infty$)

$$\sum_{d \le D} \frac{1}{d} = \log D + \gamma + O\Big(\frac{1}{D}\Big).$$

These two, together with γ & γ_1 definitions (see §2), suffice (recalling $L = [\log N]$, here) to get $\forall \varepsilon > 0$, say,

$$i) \qquad \sum_{q \le \frac{x}{N^{1/3}}} \frac{\mathbf{d}(q)}{q} = \frac{1}{2}\log^2 x - \frac{L}{3}\log x + \frac{L^2}{18} + 2\gamma \log x - \frac{2}{3}\gamma L + (\gamma^2 + 2\gamma_1) + O_\varepsilon(N^{\varepsilon - 1/3}).$$

$$ii) \qquad \sum_{d_1 \le N^{1/3}} \frac{1}{d_1} \sum_{d_2 \le \frac{x}{d_1 N^{1/3}}} \frac{1}{d_2} = \frac{L}{3}\log x + \gamma \log x + \gamma^2 - \frac{L^2}{9} - \frac{L^2}{18} + \gamma_1 + O_\varepsilon\Big(\frac{1}{N^{1/3-\varepsilon}}\Big).$$

$$iii) \qquad \Big(\sum_{d \le N^{1/3}} \frac{1}{d} \Big)^2 = \frac{L^2}{9} + \frac{2\gamma}{3}L + \gamma^2 + O_\varepsilon\Big(\frac{1}{N^{1/3-\varepsilon}}\Big).$$

Summing up these $i)$ (the only one requiring Amitsur-Tull bounds), $ii)$ and $iii)$ (the two elementary ones),

$$\frac{\widetilde{M}_3(x,h)}{h} = \sum_{q \le \frac{x}{N^{1/3}}} \frac{g_3(q)}{q} = \frac{1}{2}\log^2 x + 3\gamma \log x + (3\gamma^2 + 3\gamma_1) + O_\varepsilon\left(\frac{N^\varepsilon}{N^{1/3}}\right) = Q_2(\log x) + O_\varepsilon\left(\frac{N^\varepsilon}{N^{1/3}}\right) =$$

$$= \frac{M_3(x,h)}{h} + O_\varepsilon\left(\frac{N^\varepsilon}{N^{1/3}}\right),$$

whence $(*)$.

In perfect analogy with $i)$, $ii)$ and $iii)$, we may calculate the main term of, say,

$$\sum_{q\le Q} g_3(q) \stackrel{def}{=} \sum_{q\le Q} a_3(q) + O(\sqrt{Q}),$$

namely, in an elementary way (Dirichlet hyperbola again!), so to define:

$$g^{av} \stackrel{def}{=} g_3 - a_3 \quad\Rightarrow\quad \Big|\sum_{q\le Q} g^{av}(q)\Big| \ll \sqrt{Q}$$

and this can be applied in the following ranges of q, not depending on x. Then, the mean-squares of, say, $a_3 * \mathbf{1}$ may be treated with Lemma 0 and Lemma 2, since these a_3 are linear combinations of log-powers; this is reminiscent of the same method of proof of Theorem 1.1 [C2], for the coefficients that, actually, give the Selberg integral mean-values (compare the above). In fact, as stated in the sketch above, point 1),

$$\widetilde{J}_3(N,h) \lll \sum_{x\sim N}\Big|\sum_{q\le x/[N^{1/3}]} g_3(q)\widetilde{\chi}_q(x,h)\Big|^2 + N^{1/3}h^2$$

follows from $(*)$ above and, say, for $M := [N^{1/3}]$ here (small errors w.r.t. the previous); then,

$$\sum_{x\sim N}\Big|\sum_{q\le x/M} g_3(q)\widetilde{\chi}_q(x,h)\Big|^2 \ll \sum_{x\sim N}\Big|\sum_{q\le 2N/M} g_3(q)\widetilde{\chi}_q(x,h)\Big|^2 + \sum_{x\sim N}\Big|\sum_{x/M<q\le 2N/M} g_3(q)\widetilde{\chi}_q(x,h)\Big|^2 \lll$$

$$\lll \sum_{x\sim N}\Big|\sum_{q\le 2N/M} g^{av}(q)\widetilde{\chi}_q(x,h)\Big|^2 + \sum_{x\sim N}\Big|\sum_{q\le 2N/M} a_3(q)\widetilde{\chi}_q(x,h)\Big|^2 + \sum_{x\sim N}\Big|\sum_{x/M<q\le 2N/M} g_3(q)\widetilde{\chi}_q(x,h)\Big|^2$$

$$\lll Nh + \frac{N}{M}h^2 + \sum_{x\sim N}\Big|\sum_{x/M<q\le 2N/M} g_3(q)\widetilde{\chi}_q(x,h)\Big|^2$$

where in the first integral $Q := 2N/M \ll N^{2/3}$ gives $\lll Nh + N^{2/3}h^2$, applying the Proposition to $g = g^{av}$ as above, instead of g_3 (so the sum up to Q is negligible) and (as quoted above) we apply Lemmas 0,2 for a_3 mean-square; while the other is treated trivially, recall $Q = 2N/M$, from $g_3 \ge 0$ and $C_h \ge 0$

$$\sum_{x/M<q\le Q} g_3(q)\widetilde{\chi}_q(x,h) \ll \sum_{x/M<q\le Q} g_3(q) \sum_{n\equiv 0(q)} C_h(n-x) + h \sum_{x/M<q\le Q} \frac{g_3(q)}{q} \ll$$

$$\ll \sum_{Q/2<q\le Q} g_3(q) \sum_{n\equiv 0(q)} C_h(n-x) + h \sum_{Q/2<q\le Q} \frac{g_3(q)}{q} \ll \Big|\sum_{Q/2<q\le Q} g_3(q)\widetilde{\chi}_q(x,h)\Big| + h \sum_{Q/2<q\le Q} \frac{g_3(q)}{q}$$

and, for the new term with $\widetilde{\chi}_q(x,h)$ we apply again the Proposition, getting same remainders as above; then, we are left with the trivial estimate of the new mean-value, namely

$$h \sum_{Q/2<q\le Q} \frac{g_3(q)}{q} \ll h \sum_{Q/2<q\le Q} \frac{\mathbf{d}(q)}{q} \ll \frac{h}{Q} \sum_{Q/2<q\le Q} \mathbf{d}(q) \ll hL,$$

from the elementary formula for the divisor function $\mathbf{d}(n)$, or even Amitsur's formula & Tull remainders.

Of course, the Proposition gives a much better upper bound than the present "trivial" one, which in mean-square "leads", say, our bounds, giving finally $\ll Nh^2L^2$. $\square$

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